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Optimal Control of Switched Hybrid Systems: A Brief Survey

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Optimal Control of Switched Hybrid Systems: A Brief Survey

Abstract

This paper surveys recent results in the field of optimal control of hybrid and switched systems. We summarize recent results that use different problem formulations and then explore the underlying relations among them. Based on the types of switching, we focus on two important classes of problems: internally forced switching (IFS) problems and externally forced switching (EFS) problems. For IFS problems, we focus on the optimal control techniques for piecewise affine systems. For EFS problems, the optimal control of autonomous and non-autonomous switched systems are investigated. Optimization methods found in the literature are discussed.

I. Introduction

Hybrid systems are heterogeneous dynamical systems which involve both continuous models, describing the physical part and discrete event models, describing the software and logical behavior [1]. Continuous models consist of time-driven continuous variable dynamics, which are usually described by differential or difference equations; discrete event models contain event-driven discrete logic dynamics, often described by finite-state machines or Petri nets. As such, hybrid systems theory combines ideas originating in the computer science and software engineering disciplines on one hand, and systems theory and control engineering on the other. This mixed character explains the terminology “hybrid systems”. Hybrid systems have been identified in a wide variety of applications: in the control of mechanical systems, process control, automotive industry, power systems, aircraft and traffic control, among many other fields. More detailed historical review of hybrid systems can be found in [2], [3], [4], and more recent reviews are given in [5], [6]. The published books on hybrid systems include [7], [8], [9], [10], [11], [12], [13].

Switched systems are a particular class of hybrid systems consisting of several subsystems and a switching law specifying the active subsystems at each time instant. Although switched system models are relatively simple and straightforward, this system class exhibits several typical behaviors of hybrid dynamical systems. For the recent results on stability and stabilization of switched systems, one can refer to the survey [14] and the references therein.

In addition to stability and stabilization issues, the problem of determining optimal control laws for hybrid systems and in particular for switched systems, has been extensively investigated in recent years. It has attracted researchers from various fields in science and engineering. There are both theoretical and computational results. The available theoretical results usually extend the classical maximum principle or the dynamic programming approach to switched systems [15], [16], [17]. The computational results take advantage of various optimization techniques and high-speed computers to develop efficient numerical methods for the optimal control of switched systems [18], [19], [20], [21], [22], [23], [24], [25], [26].

This paper surveys recent progress in computational methods for optimal control problems of switched systems. Such problems are difficult to solve due to switching of subsystem dynamics. The past decade has seen some breakthroughs in theoretical results as well as the development of efficient computational methods. However there are no theoretical or computational results applicable to general optimal control problems for all types of switched systems. The existing literature results are often based on different models and differ in problem formulation and approaches. Here, we focus on summarizing recent results that have been used different problem formulations and then exploring the underlying relations among them.

The paper is organized as follows. In Section II, a brief overview of theoretical results on optimal control of hybrid systems is presented and the general optimal control problem formulation of switched systems is given. Section III reviews existing optimal control methodologies for switched systems with internally forced switching (IFS). Optimal control problems of piecewise affine systems are discussed. Section IV focuses on the results on optimal control of switched systems with externally forced switching (EFS). To address the role of EFS, we first present the current results for autonomous switched systems. Then the optimization techniques for non-autonomous switched systems are described. A brief summary in Section V concludes the paper.

II. Optimal control problems in hybrid and switched dynamical systems

A. *Hybrid systems*

Several approaches to determine control laws for hybrid systems have been reported in the control and computer science literature. Numerous results on necessary conditions for optimality have appeared for a variety of models of hybrid systems [27], [28], [16], [15], [17], [29], [30], [31], [32], [33], [34], [35]. However, most of the results consider general problem settings and it is not always possible to develop tractable algorithms to numerically compute the optimal solution.

For continuous-time hybrid systems, [27] compared several algorithms for optimal control, while [28] discussed general necessary conditions for the existence of optimal control laws for hybrid systems by using dynamic programming. They established a general hybrid framework for the optimal control problem, proved the existence of optimal (relaxed or chattering) controls and near-optimal (precise or nonchattering) controls, and derived generalized quasi-variational inequalities (GQVI's) that the associated value function is expected to satisfy.

Necessary optimality conditions for trajectories of hybrid systems were derived using the maximum principle by [17] and [16], who considered a fixed sequence of finite length. Several versions of hybrid maximum principles are proposed. A similar approach was used by [15], who considered both autonomous and controlled switchings with linear quadratic cost functionals.

[30], [31] used convex dynamic programming (CDP) to approximate hybrid optimal control laws and to compute lower and upper bounds of the optimal costs. The case of piecewise-affine systems was discussed by [29]. For determining the optimal feedback control law these techniques require the discretization of the state space in order to solve the corresponding Hamilton-Jacobi-Bellman equations.

[32] considered a finite-time hybrid optimal control problem and gave necessary optimality conditions for a fixed sequence of modes using the maximum principle. In [33] these results were extended to non-fixed sequences by using a suboptimal result based on the Hamming distance permutations of an initial given sequence. Finally, in [34], [35], a feedback law for a finite time LQR problem was derived by integrating the computation of the optimality zones into the hybrid maximum principle algorithms class.

A special class of hybrid systems motivated by the structure of manufacturing systems were considered by [36], [37], [38], [39], [40], [41]. The hybrid system model considered extends event-driven models to include time-driven dynamics where the event-driven dynamics follow the model in queueing theory. Due to the non-differentiability in event-driven processes, the problem to find the optimal control input is a non-smooth optimization problem. The results showed the necessary condition for an optimal control input and properties of the optimal sample paths, which can be used to identity the non-smooth part in the solution.

B. Switched systems

In order to find ways to numerically compute the optimal control in hybrid systems, many researchers have been focusing on switched systems. A switched system may be obtained from a hybrid system by simplifying the details of the discrete behavior to switching patterns from a certain class, which usually represent discontinuity in vector fields. For simplicity, it is often assumed that all subsystems are in the same state space, e.g. $\mathbb{R}^n$.

The mathematical representation of switched systems is based on the state-space model.

Definition 1. A switched system is described by a collection of indexed differential (or difference) equations

$$\dot{x}(t) = f_{q(t)}(x(t), u(t)),\ x(0) = x_0 \tag{1}$$

$$y(t) = g_{q(t)}(x(t), u(t)). \tag{2}$$

where the input is $u \in \mathbb{R}^m$, $x \in \mathbb{R}^n$ is the continuous state vector and $q(t) \in \{1, 2, \cdots, M\} \triangleq Q$.

The discrete event dynamics are modeled by a switching signal, which is usually described as a timed sequence, $\sigma = ((t_0, i_0), (t_1, i_1), \cdots, (t_K, i_K))$ where $0 \le K < \infty$, $t_0 \le t_1 \le \cdots \le t_K \le t_f$, and $i_k \in Q$ for $0 \le k \le K$. A timed sequence directs when switching would happen and what the successive subsystem will be. According to the nature of switching signals, switched systems may be classified into two classes: switched systems with *externally forced switching* (EFS) or *internally forced switching* (IFS). As shown in Fig. 1, the switching signal σ of EFS is an exogenous input to the system, as the continuous input u. Hence one has freedom to choose a specific switching signal or a class of switching signals of interests to study the behavior of systems. The switching signal σ of IFS is based on the information of the state x and the current mode q. In general, the switching law can be a function of t, x, u and q, as shown in Fig. 2.

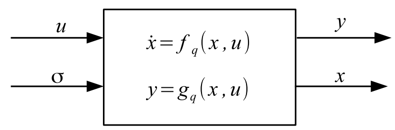

Figure 1. Switched systems with *externally forced switching* (EFS)

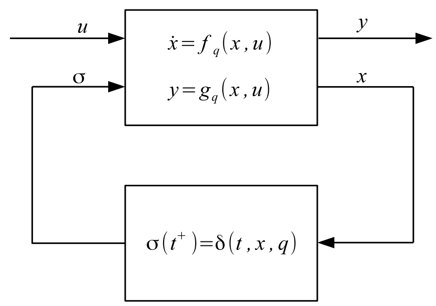

Figure 2. Switched system with *internally forced switching* (IFS)

Remark 2. For a switched system with EFS, the exogenous control input is a pair (σ, u). For a switched system with IFS, the exogenous control input is u. The switching sequence σ is generated implicitly based on the evolution of x and u.

Remark 3. A variety of switched system models can be derived from Eq. 1-2. Based on the different dynamics of subsystems (characterized by indexed differential or difference equations), we have *continuous-time (discrete-time) switched systems,* if the subsystems are continuous (discrete) time systems, or *switched linear (nonlinear) systems,* if subsystems are linear (nonlinear) systems. If the continuous control input u is absent from the model, we call it an *autonomous switched system*. The continuous state x usually does not exhibit jumps at switching instants (i.e. $x(t_i^+) \neq x(t_i^-)$). However, we note that some methods reported here can be extended to problems with jumps.

C. Two optimal control problems of switched systems

Although in general optimal control problems would be formulated for switched systems with both EFS and IFS, results would be difficult to obtain. Therefore we focus on two important classes of problems which can be solved individually, namely, optimal control problem with EFS only (EFS Problems), and problems for systems with IFS only (IFS problem). Most of the literature in this paper addresses one of these problems.

Problem 4 (EFS problem)**.** [18] Consider a switched system with EFS. Find an admissible control pair

(σ, u) (u is piecewise continuous) such that x departs from a given initial state $x(t_0) = x_0$ at the given initial time t_0 and meets the terminal manifold defined by $\psi\left(x\left(t_f\right), t_f\right) = 0$ where ψ is a vector function and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \sum_{1\leq k\leq K} \gamma(x(t_k), i_{k-1}, i_k) \tag{3}$$

is minimized (here K is the number of switchings in σ).

Problem 5 (IFS problem)**.** [18] Consider a switched system with IFS. Find an admissible control $u(t)$ such that x departs from a given initial state $x(t_0) = x_0$ at the given initial time t_0 and meets he terminal manifold defined by $\psi\left(x\left(t_f\right), t_f\right) = 0$ where ψ is a vector function and

$$J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt + \sum_{1\leq k\leq K} \gamma(x(t_k), i_{k-1}, i_k) \tag{4}$$

is minimized (here K is the number of switchings in σ_I).

Remark 6. Problem 4 and Problem 5 are formulated as general optimal control problems with terminal cost ψ, running cost $\int_{t_0}^{t_f} L dt$, and switching cost γ. Subject to applicability and solvability, different problems may adopt the specific forms of the terminal, running and switching cost or drop certain cost terms.

Remark 7. The main difference between the two problems is whether switching is an exogenous input or autonomously generated. For EFS problem, one needs to optimize both the continuous control input u and the switching signal σ, which are strongly coupled in the optimization process. On the other hand, the difficulty in IFS problem is that the switching depends on the specific initial state and the control $u(t)$ (e.g. switching surfaces characterized by functions of states and input) and cannot be explicitly determined unless a specific control input is given.

III. Optimal control for switched systems with IFS

IFS problems concentrate on finding a continuous control input u to minimize the cost function when switching is autonomous. The difficulty of this problem is that the switching instants can depend on the continuous control input u in a very complicated way. Among the models of switched systems, piecewise affine system models are suitable for IFS problem since the switchings are implicitly determined by the partition of the state and input spaces.

A piecewise affine (PWA) system is defined by partitioning the state space into polyhedral regions, and associating with each region a different linear state-update equation

$$x(t+1) = A_i(t) + B_i u(t) + f_i \tag{5}$$

$$\text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathcal{X}_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : H_i x + J_i u \leq K_i \right\}$$

where $x \in \mathbb{R}^{n_c} \times \{0,1\}^{n_l}$, $u \in \mathbb{R}^{m_c} \times \{0,1\}^{m_l}$, $\{\mathcal{X}_i\}_{i=0}^{s-1}$ is a polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$, $n \triangleq n_c + n_l$, $m \triangleq m_c + m_l$. PWA systems can model a large number of physical processes, such as systems with static nonlinearities, and can approximate nonlinear dynamics via multiple linearizations at different operating points [42], [43]. Details and recent results on PWA can be found [44], [45], [46], [47], [48].

When the hard input and state constraints

$$Ex(t) + Lu(t) \leq M \tag{6}$$

are introduced, the constrained PWA system (CPWA) can be obtained as in Eq. 7

$$x\,(t+1) = A_i x(t) + B_i u(t) + f_i \tag{7}$$

$$\text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \triangleq \left\{ \begin{bmatrix} x \\ u \end{bmatrix} : \tilde{H}_i x + \tilde{J}_i u \leq \tilde{K}_i \right\}$$

where $\left\{\tilde{\mathcal{X}}_i\right\}_{i=0}^{s-1}$ is the new polyhedral partition of the sets of state+input space $\mathbb{R}^{n+m}$ by intersecting the polyhedrons $\mathcal{X}_i$ in Eq. 5 with the polyhedron described by Eq. 6.

After defining the cost function

$$J\left(U_0^{T-1}, x(0)\right) \;\triangleq\; \sum_{k=0}^{T-1} (\|Qx(k)\|_p + \|Ru(k)\|_p) + \|Px(T)\|_p \tag{8}$$

the finite-time optimal control problem (FTCOC) is formulated as in Eq. 9-10

$$J^*(x(0)) \triangleq \min_{\{U_0^{T-1}\}} J(U_0^{T-1}, x(0)) \tag{9}$$

$$s.t. \begin{cases} x(t+1) = A_i x(t) + B_i u(t) + f_i \\ \qquad \text{if } \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \tilde{\mathcal{X}}_i \end{cases} \tag{10}$$

where the column vector $U_0^{T-1} \triangleq [u'(0), \ldots, u(T-1)']' \in R^{mT}$, is the optimization vector and T is the time horizon. In Eq. 8, $\|Qx\|_p = x'Qx$ for $p = 2$ and $\|Qx\|_p = \|Qx\|_{1,\infty}$ for $p = 1, \infty$, where $R = R' \succ 0$, $Q = Q'$, $P = P' \succ 0$ if $p = 2$ and Q, R, P non-singular if $p = \infty$or $p = 1$.

The main results on FTCOC can be found in [49], [50], [21], [51], [52], [53], [54], [55], [56], [57], [58], [59], [60]. It has been proved that the closed form of the state-feedback solution to the finite time optimal control problem based on quadratic or linear norms performance criteria, is a time-varying piecewise affine feedback control law. Moreover, two computational methods are provided to numerically find the optimal solution.

One way is describing the PWA system by a set of inequalities with integer variables as the system switches between the different dynamics. An appropriate modeling framework for such class of systems is the mixed logical dynamic framework. Mixed logical dynamical (MLD) systems are computationally oriented representations of hybrid systems that consist of a collection of linear difference equations involving both real and Boolean (i.e. 1 or 0) variables, subject to a set of linear inequalities. Details and recent results on MLD can be found in [61], [62].

After transforming the PWA system to its equivalent MLD system, Eq. 8-10 can be rewritten as:

$$\min_{\{U_0^{T-1}\}} J\left(U_0^{T-1}, x(0)\right) \triangleq \sum_{k=0}^{T-1}(\|Qx(k)\|_p + \|Ru(k)\|_p) + \|Px(T)\|_p \tag{11}$$

$$\text{s.t.} \begin{cases} x(k+1) = Ax(k) + B_u u(k) + B_\delta \delta(k) + B_z z(t), \\ E_\delta \delta(k) + E_z z(k) \leq E_u u(t) + E_x x(t) + E \end{cases} \tag{12}$$

The optimal control problem in Eq. 11-12 can be formulated as a Mixed Integer Quadratic Program (MIQP) when the squared Euclidean norm $p = 2$ is used [61], or as a Mixed Integer Linear Program (MILP), when $p = \infty$ or $p = 1$ [53]. In addition, multiparametric programming can be used to efficiently compute the explicit form of the optimal state-feedback control law [53], [55].

Another method of combining a dynamic programming strategy with a multi-parametric program solver is proposed in [59], [60]. The dynamic program is of the following form

$$J_j^*(x(j)) \triangleq \min_{u(j)} \|Qx(j)\|_p + \|Ru(j)\|_p + J_{j+1}^*(f_{PWA}(x(j), u(j))) \tag{13}$$

$$\text{s.t. } f_{PWA}(x(j), u(j)) \in \mathcal{X}^{j+1} \tag{14}$$

for $j = T-1, \ldots, 0$, with boundary conditions

$$X^T = X^f, \text{ and} \tag{15}$$

$$J_T^*(x(T)) = \|Px(T)\|_p \tag{16}$$

where

$$X^j = \left\{x \in \mathbb{R}^n | \exists u,\ f_{PWA}(x, u) \in \mathcal{X}^{j+1}\right\} \tag{17}$$

is the set of all initial states for which the problem Eq. 13-14 is feasible.

Depending on the norm used in the cost function Eq. 13, the algorithm based on dynamic programming recursion and multiparametric quadratic solvers is used when $p = 2$ [59], [60]. Similarly, when $p = 1$ or $p = \infty$, the algorithm based on dynamic programming recursion and multiparametric linear program solvers is shown in [49], [21], [60]. Compared with the algorithm based on MIP, the dynamic programming algorithm is more efficient and less complex due to fewer underlying inequality constraints. Also, the dynamic programming algorithm can be used to approximate infinite time horizon solutions through finite time horizon solutions. Recent work [51], [52] showed how to exploit the underlying geometric structure of the optimization problem with a linear performance index in order to significantly improve the efficiency of the off-line computations. [57], [58] studied the constrained finite-time optimal control problem of discrete-time nonlinear systems using algebraic geometry methods.

IV. Optimal control for switched systems with EFS

A. Autonomous switched systems

To address the role of EFS in optimal control problem, [63], [64], [65], [66], [67], [68], [69], [70], [71], [22], [72], [73], [74], [75], [76], [77], [78], [23], [79], [80], [81], [82], [83], [84], [85], [24] considered autonomous switched systems (when continuous control input u is absent) and solely focus on deriving the optimal switching signal. Recall that a switching signal consists of switching times and ordered indices of systems. Even when the continuous control input u is absent, finding an optimal switching signal can be challenging.

We first consider an *autonomous switched nonlinear system*

$$\dot{x} = f_i(x(t)), \tag{18}$$

for all $t \in [\tau_i, \tau_{i+1})$, and for every $i \in \{0, \ldots, N\}$, with the given initial condition $x(0) = x_0$. τ_i are a sequence of switching times with $\tau_0 = 0$ and $\tau_{N+1} = T$. T is fixed. Let $L : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function. The cost function J is defined by

$$J = \int_0^T L(x(t))dt. \tag{19}$$

As a starting point, a simpler version of the problem was considered by [79], [81], where the switching sequence (i.e. the ordered indices) is *fixed* for a switched nonlinear system. Therefore, the controlled parameter to optimize the cost function is just a sequence of switching times $\bar{\tau} := (\tau_1, \ldots, \tau_N)^T$. In [79], [81], a simple formula for the gradient $\nabla J(\bar{\tau})$ was derived, and it was directly used in gradient-descent algorithms. Later, various practical issues, such as state estimation [72], [80] and on-line computation [85], [24], [83], [84], were addressed when finding the optimal switching times. [72], [80] presented the algorithms to solve the problem for autonomous switched systems when the state of the system is only partially known through the outputs. A method was presented to guarantee that the current switch-time estimates remain optimal as the state estimates evolve in a computationally feasible manner. Inspired by this work, more results [85], [24], [83], [84] appeared on the research of on-line optimization of switched systems. The need for real-time on-line algorithms typically arises when complete information about the system is not available but the algorithm can acquire partial information about it in real time. If the *state jumps* (i.e. $x(\tau_i^+) \neq x(\tau_i^-)$) appear, [86] showed that an approximate solution for this optimal control problem can be computed by solving a sequence of conventional dynamic optimization problem. This approach can reduce the excessive switching between subsystems by merging two or more switching times into one.

For the problem of finding the optimal switching sequence and switching times *jointly*, various approaches are proposed. Although different approaches have their specific switched systems models and cost functions, they can be classified into two kinds of methodologies: two-level optimization and embedding transformation.

1) Two-level optimization: A two-level optimization algorithm consists of two separate levels of optimization. At the lower level, the algorithm assumes a fixed switching sequence and minimizes the cost functional with respect to the switching times. At the upper level, it updates the switching sequence to minimize the cost function. The whole optimization process is based on alternating between these two procedures.

For autonomous switched *nonlinear* system (Eq. 18), [76], [23] proposed a framework consisting of the switching times and the switching sequence as variables to be optimized. At the lower level, the algorithm minimizes the cost functional with respect to the switching times. At the upper level, it updates the switching sequence by inserting two switching points at some time with a subsystem index between them. Later, a notion of local optimality and a suitable concept of convergence were defined in [77] to devise a provably convergent optimization algorithm.

For autonomous switched *linear* systems, [65], [64], [63], [66], [67], [68], [69], [70], [71], [22] consider s*witched affine systems with state jumps*, defined in Eq. 20 and 21

$$\dot{x}(t) = A_{i(t)}x(t) + f_{i(t)},\ i(t) \in \mathcal{S}, \tag{20}$$

$$x(t^+) = J_{i,j}x(t^-)\ if\ i(t^-) = i,\ i(t^+) = j. \tag{21}$$

Eq. 21 models a jump condition: whenever at time t a switch from mode $i(t^-) = i$ to mode $i(t^+) = j$ occurs, the state jumps from $x(t^-)$ to $x(t+) = J_{i,j}x(t^-)$, where $J_{i,j} \in \mathbb{R}^{n\times n}$ are constant matrices. The main objective is to solve the optimal control problem

$$V_N^* \ \triangleq\ \min_{\mathcal{I},\mathcal{T}} \int_0^\infty x'(t)Q_{i(t)}x(t)dt + \sum_{k=1}^{N} H_{i_{k-1},i_k} \tag{22}$$

$$\text{s.t.} \begin{cases} \dot{x}(t) = A_{i(t)}x(t) + f_{i(t)} \\ x(0) = x_0 \\ i(t) = i_k \in \mathcal{S}, \tau_k \le t \le \tau_{k+1},\ k = 0,\ldots,N \\ 0 = \tau_0 \le \tau_1 \le \ldots \le \tau_N < \tau_{N+1} = +\infty \\ x(\tau_k^+) = J_{i_{k-1},i_k} \ldots J_{i_{h-1},i_h} x(\tau_h^-) \end{cases} \tag{23}$$

where Q_i are positive semi-definite matrices, and x_0 is the initial state of the system. The cost Eq. 22 consists of two components: a quadratic cost that depends on the time evolution (the integral) and a cost that depends on the switches (the sum), where $H_{i,j} \ge 0$, $i, j \in \mathcal{S}$, is the cost for commuting from mode i to mode j, with $H_{i,i} = 0$, $\forall i \in \mathcal{S}$. In this optimization problem there are two types of decision variables: a finite sequence of switching times $\mathcal{T} \triangleq \{\tau_1,\ldots,\tau_N\}$ and a finite sequence of modes $\mathcal{I} \triangleq \{i_0,\ldots,i_N\}$.

To solve such an optimal control problem, a *master-slave procedure* (MSP), which can be viewed as a two-level optimization algorithm, is applied. The "master" procedure is based on mixed-integer quadratic programming (MIQP) and finds an optimal switching sequence for a given initial state, assuming the switching instants are known. The "slave" procedure is based on the construction of the switching regions and finds the optimal switching instants, assuming the mode sequence is known. The results in [64], [65], [69], [70] showed that it is possible to numerically compute a region of the state space in which an optimal control switch should occur. As a generalization of the slave procedure, a *switching table procedure* (STP) was presented in [63], [22]. STP is based on dynamic programming ideas and allows one to avoid the explosion of the computational burden as the number of switching sequences increases. It relies on the construction of switching tables which specify when the switching should occur and what the next mode should be. Moreover, [66], [68], [67], [71] showed that STP can be applied to other hybrid system frameworks.

2) Embedding transformation: The basic idea of embedding transformation is that one can transform switched systems to a larger family of systems where the switching sequence, number of switchings and switching instants are embedded as decision variables.Then the various conventional optimization techniques can be utilized.

The application of this transformation to switched *nonlinear* systems was considered in [87]. The original problem was first transformed into a continuous polynomial system and then the method of moments was applied to find the optimal switching signal. The necessary and sufficient condition for optimality was obtained and the example showed that existing numerical methods in convex programming can be used to solve the problem. For switched *linear* systems, [88] used a different embedding transformation. It showed that the optimal solution can be determined by solving the two-point boundary value problem after applying Pontryagin's Minimum Principle.

B. Non-autonomous switched systems

In order to solve the EFS problem completely , it is necessary to consider the continuous input u together with switching times and sequences. Similar to the two-level optimization for autonomous switched systems, several two-stage optimization techniques are adapted to find the optimal solutions for non-autonomous switched systems. [89], [90], [91], [92], [93], [94], [18], [95], [96], [97], [19], [98] considered the optimal control problem of seeking both the optimal control switching times and continuous control input for *continuous-time switched systems*. *Constrained switched nonlinear systems* were considered by [99], [100]. A different embedding transformation technique was investigated by [20], [101]. For *discrete-time switched linear systems* with quadratic cost functions (i.e. switching linear quadratic regulator (LQR) problem), [102] and [26] pointed out that the optimal switching and continuous control input can be calculated by solving a set of difference Riccati Equations (DRE). They focused on the techniques of reducing the solution size which will grow exponentially as the time evolves.

1) Two-stage optimization: [90], [91], [18], [19] proposed a computational method based on a two-stage optimization method and prove that it solves the EFS problem assuming that a pre-specified switching sequence is given. The method is based on the fact that the following equation holds:

$$\begin{aligned} &\min_{(\sigma,u)\in\Omega} J\left(\sigma,u\right) = J\left(\sigma^*,u^*\right) \\ &= \min_{\sigma\in\{\sigma\mid\exists u,(\sigma,u)\in\Omega\}} \; \min_{u\in\{u\mid\exists\sigma,(\sigma,u)\in\Omega\}} J\left(\sigma,u\right) \end{aligned} \tag{24}$$

where Ω is the set in which (σ, u) is admissible and $J = \psi(x(t_f)) + \int_{t_0}^{t_f} L(x(t), u(t))dt$. It is easy to see that the right hand side of Eq. 24 requires twice the minimization processes. This implies that the following two stage optimization can be applied.

Stage 1 *(a) Fix the total number of switchings to be K and the sequence of active subsystems. Let the minimum value of J with respect to u be a function of the K switching instants, i.e., $J_1 = J_1(t_1, \cdots t_K)$ for $K \geq 0$ $(t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f)$. Find the form of J_1.*

Stage 1 *(b) Minimize $J_1$with respect to $t_{1,} \cdots, t_K$. i.e.*

$$\min_{\hat{t}} J_1(\hat{t})$$

subject to $\hat{t} \in T$, where $T \triangleq \{\hat{t} = (t_1, \cdots, t_K)^T | t_0 \leq t_1 \leq \cdots \leq t_K \leq t_f\}$.

Stage 2 *(a) Vary the sequence of active subsystems to find an optimal solution under K switchings.*

Stage 2 *(b) Vary the value of K to find an optimal solution, eventually for Problem 4.*

It can be seen that the stage 1 and stage 2 are (relatively) decoupled problems and therefore can be solved separately. Since the stage 2 can be formulated as another search problem, [89], [91] assumed the switching sequence is prefixed and focus on the stage 1 optimization. The conceptual computational algorithm for stage 1 optimization is as follows.

1) *Set the iteration index $j = 0$. Choose an initial $\hat{t}^j$.*
2) *By solving an optimal control problem, find $J_1(\hat{t}^j)$.*
3) *Find $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ (and $(\partial^2 J_1/\partial \hat{t}^2)(\hat{t}^j)$ if second-order method is to be used).*
4) *Use some feasible direction method to update $\hat{t}^j$ to be $\hat{t}^{j+1} = \hat{t}^j + \alpha^j d\hat{t}^j$ (here $d\hat{t}^j$ is formed by using the gradient information of J_1, the step-size α^j can be chosen using some step-size rule, e.g., Armijo's rule). Set the iteration index $j = j + 1$.*
5) *Repeat Steps (2), (3), (4) and (5), until a pre-specified termination condition is satisfied (e.g., the norm of the projection of $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ on any feasible direction is smaller than a given small number ϵ).*

In fact, Step 2 poses an obstacle for the usage of the algorithm because the values of the derivatives $(\partial J_1/\partial \hat{t})(\hat{t}^j)$ and $(\partial^2 J_1/\partial \hat{t}^2)(\hat{t}^j)$ are not readily available. In [89], [91], a method based on direct differentiations of the value function was proposed to approximate the values of the derivatives. In [19], [103], based on an equivalent problem formulation, accurate values of the derivatives were obtained by solving a two point boundary value differential algebraic equation (DAE).

A similar multi-stage optimization mechanism can be found in [99], [100]. The innovation is that the proposed method aims to find the optimal continuous control input as well as the full information of switching signal (switching sequence and time instants), without assuming the switching sequence is prefixed. The switched system model used is *constrained switched nonlinear systems*. By "constrained switched nonlinear systems", it is meant that the state x is constrained to a set described by

$$x(t) \in \{x \in \mathbb{R}^n | h_j(x) \leq 0, \ j = 1, ..., N\}. \tag{25}$$

for all time and the cost function is defined in Eq. 3 but without the switching cost term. Based on this model, [99] developed a *bi-level hierarchical algorithm* that divides the problem into two nonlinear constrained optimization problems, as presented below.

Stage 1 *Given a switching sequence σ, calculate the optimal switching time sequence s and the optimal control input u.*

Stage 2 *Calculate a new sequence $\tilde{\sigma}$, which is the result of the insertion of a new switching into the original sequence σ. Repeat Stage 1 using $\tilde{\sigma}$.*

At the lower level, the algorithm assumes a fixed modal sequence and determines the optimal mode duration and optimal continuous input. At the higher level, the algorithm employs a single mode insertion technique to construct a new lower cost sequence. The search for all possible switching sequences, which could grow exponentially after each iteration, can be avoided by a single mode insertion technique. An improved algorithm was presented recently [100], in which new features are implemented to overcome certain shortcomings of the original algorithm.

A recent result [104], considered the optimal control problem of *discrete-time nonlinear switched systems*. It shows that the problem can be numerically solved by dividing the original control problem into two sub-problems. The optimal continuous control input can be obtained from the first sub-problem for a given switching sequence. The second sub-problem will search for the optimal switching sequence through the discrete filled function method. The global optimal solutions (for most cases) can be found by iteratively solving the two sub-problems.

2) Embedding transformation: [20], [101] considered solving the optimal control problem for continuous-time switched nonlinear systems through embedding. The switched system is first embedded into a larger family of continuous systems namely

$$x(t) = \sum_{i=1}^{N} v_i(t) f_i(x(t), u_i(t)) \tag{26}$$

where $v_i \in [0,1]$ and $\sum_{i=1}^{N} v_i(t) = 1$.Then the embedded system can be solved using conventional optimal control techniques. By adopting such problem transformation, there is no need to make any assumptions about the number of switches nor about the mode sequence at the beginning of the optimization. The theoretical results in [20] showed that the set of trajectories of the switching system is dense in the set of trajectories of the embedded system. Furthermore, the results also imply that if one solves the embedded optimal control problem and obtains a solution, either the solution is the solution of the original problem, or suboptimal solutions can be constructed. Recently, [101] further explored the possible numerical nonlinear programming techniques under this framework. It showed that sequential quadratic programming (SQP) can be utilized to reduce the computational complexity introduced by mixed integer programming (MIP). The effectiveness of the proposed approach was demonstrated through several examples.

3) Switching LQR problem: [26] and [102], [105], [106], [25] studied the optimal control and scheduling problem of discrete-time switched linear systems. The model considered is

$$x\,(k+1) = A_{j(k)}x(k) + B_{j(k)}u(k). \tag{27}$$

Switching between subsystems is described by the switching index $j(k)$ which is subject to control. Further, the cost function is in the quadratic form

$$J_N(u,j) \;\; = \;\; \sum_{k=0}^{N-1} \left(x^T(k)Q_{j(k)}x(i) + u^T(k)R_{j(k)}u(i)\right) + x^T(N)Q_f x(N) \tag{28}$$

where Q_f and Q_j are symmetric positive semi-definite and R_j is symmetric positive definite. The goal is to find a sequence of control inputs $\{u(k)\}_{k=1}^{N}$ and switching sequence $\{j(k)\}_{k=1}^{N}$ such that the cost function J_N is minimized. The main results in [26], [102] showed that the explicit feedback form of the optimal control input can be obtained by solving a set of difference Riccati Equations (DRE). The resulting optimal control input $u(k)$ is in piecewise linear state feedback form and the switching sequence and times are obtained through dynamic programming. It is worth pointing out that both [102] and [26] faced the problem that the size of the positive semi-definite matrix set will grow exponentially as the time evolves. To reduce the solution space, [26] considered a sub-optimal cost function so that the optimal control problem can be approached by relaxed dynamic programming [107], [108]. The different approach in [105], [106] resorted to finding the minimum equivalent subset of the positive semi-definite matrix set by removing the redundant matrices. It should be noted that the optimality of the problem is not jeopardized in [105], [106]. [106] also showed that a similar algorithm can be extended to the case when sub-optimal performance is acceptable.

V. Conclusion

This survey gives a brief overview of the results on optimal control of switched systems, with emphasis on computational results recently developed. We first presented the general formulation of optimal control problems and then introduced two important problem classes based on the nature of switching. For each class of problems, the main methodologies under different assumptions and types of switched systems were summarized. For IFS problems, the computational results on piecewise affine systems were explored. For EFS problems, optimization techniques such as two-stage optimization, embedding transformation and switching LQR design were discussed. Since the research on optimal control of switched and hybrid systems is still an active area, the survey is not intended to include all the results in the literature but aims to show the essential ideas and trends in this field. We apologize for any omissions due to space limitation or not being aware of them.

It is worth to note that some software packages are available to compute the optimal control solutions of switched and hybrid systems. Multi-Parametric Toolbox (MPT) [109] is a free MATLAB toolbox for design, analysis and deployment of optimal controllers for constrained linear, nonlinear and hybrid systems. Efficiency of the code is guaranteed by the extensive library of algorithms from the field of computational geometry and multi-parametric optimization. YALMIP [110] features an intuitive and flexible modeling language for solving optimization problems. The main emphasis is on convex conic programming (linear, quadratic, second order cone and semi-definite programming), but it supports also integer programming and non-convex problems. Moreover, YALMIP can be used together with the MPT toolbox to setup and solve multiparametric optimization problems. CDP (Convex Dynamic Programming) Tool is another MATLAB toolbox developed to solve hybrid optimal control problems. The user manual can be found in [111].

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